Gamma at Reciprocals of Positive Integers

12225 [2021, 88]. *Proposed by Pakawut Jiradilok, Massachusetts Institute of Technology, Cambridge, MA, and Wijit Yangjit, University of Michigan, Ann Arbor, MI.* Let Γ denote the gamma function, defined by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ for $x > 0$.
(a) Prove that $\lceil \Gamma(1/n) \rceil = n$ for every positive integer n, where $\lceil y \rceil$ denotes the smallest integer greater than or equal to y.
(b) Find the smallest constant c such that $\Gamma(1/n) \geq n - c$ for every positive integer n.

Solution by Missouri State University Problem Solving Group, Springfield, MO. We use three facts about the gamma function: (i) $\Gamma(x+1) = x\Gamma(x)$, (ii) $\Gamma'(1) = -\gamma$, where γ is the Euler–Mascheroni constant, and (iii) the gamma function is convex on $(0, \infty)$.
(a) The equation of the line tangent to $y = \Gamma(x+1)$ at the point $(0, 1)$ is

$$y = 1 + \Gamma'(1)x = 1 - \gamma x.$$

Since the gamma function is convex, this implies that for $x > -1$,

$$\Gamma(x+1) \geq 1 - \gamma x.$$

Applying this with $x = 1/n$ yields

$$\Gamma(1/n) = n\Gamma(1/n + 1) \geq n(1 - \gamma/n) = n - \gamma.$$

Also, since $\Gamma(1) = \Gamma(2) = 1$, by convexity $\Gamma(x + 1) \leq 1$ for $0 \leq x \leq 1$. Hence

$$\Gamma(1/n) = n\Gamma(1/n + 1) \leq n.$$

Since $n - \gamma \leq \Gamma(1/n) \leq n$ and $\gamma < 1$, we conclude that $\lceil \Gamma(1/n) \rceil = n$.
(b) The solution to part **(a)** shows that γ satisfies the required condition. Now let c be any constant such that $\Gamma(1/n) \geq n - c$ for all n. We have

$$c \geq n - \Gamma(1/n) = n - n\Gamma(1/n + 1) = -\frac{\Gamma(1 + 1/n) - 1}{1/n}.$$

Letting n approach ∞ yields

$$c \geq \lim_{n\to\infty} -\frac{\Gamma(1 + 1/n) - 1}{1/n} = -\Gamma'(1) = \gamma.$$

Thus, γ is the smallest such c.

Also solved by R. A. Agnew, K. F. Andersen (Canada), P. Bracken, H. Chen, G. Fera (Italy), D. Fleischman, J.-P. Grivaux (France), J. A. Grzesik (Canada), L. Han, N. Hodges (UK), O. Kouba (Syria), O. P. Lossers (Netherlands), I. Manzur (UK) & M. Graczyk (France), R. Molinari, M. Omarjee (France), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), M. Vowe (Switzerland), T. Wiandt, J. Yan (China), L. Zhou, and the proposer.